Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

6 - 11 General Solution

Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with r(t) as given below.

6. $r(t) = \sin \alpha t + \sin \beta t$, $\omega^2 \neq \alpha^2, \beta^2$

```
Clear["Global`*"]
```

```
r[t_] := Sin[αt] + Sin[βt] /; {{ω² ≠ α²}, {ω² ≠ β²}}
```

```
DSolve[y''[t] + ω² y[t] == r[t], y[t], t]
```

$$\left\{\left\{y[t] \to C[1] \,Cos[t\,\omega] + Cos[t\,\omega] \int_1^t -\frac{r[K[1]]\,Sin[\omega\,K[1]]}{\omega}\,dK[1] + C[2]\,Sin[t\,\omega] + \left(\int_1^t \frac{Cos[\omega\,K[2]]\,r[K[2]]}{\omega}\,dK[2]\right) Sin[t\,\omega]\right\}\right\}$$

An even-numbered problem. Is the answer correct? Can't check it.

7. $r(t) = \sin t$, $\omega = 0.5, 0.9, 1.1, 1.5, 10$

```
Clear["Global`*"]
```

```
r[t_] := Sin[t]
eq1 = DSolve[y''[t] + ω² y[t] == r[t], y[t], t]
```

$$\left\{\left\{y[t] \to C[1]\,Cos[t\,\omega] + C[2]\,Sin[t\,\omega] + \frac{Cos[t\,\omega]^2\,Sin[t] + Sin[t]\,Sin[t\,\omega]^2}{-1+\omega^2}\right\}\right\}$$

$$eq2 = eq1\ /.\ \frac{Cos[t\,\omega]^2\,Sin[t] + Sin[t]\,Sin[t\,\omega]^2}{-1+\omega^2} \to \frac{Sin[t]}{-1+\omega^2}$$

$$\left\{\left\{y[t] \to C[1]\,Cos[t\,\omega] + \frac{Sin[t]}{-1+\omega^2} + C[2]\,Sin[t\,\omega]\right\}\right\}$$

Above: making a trig identity substitution by hand to conform the green cell to the text answer. The sequence of ω s makes it look like a table could be built, but not of the solution function, because the arbitrary constants blur everything. Instead the text focuses on the particle $\frac{1}{-1+\omega^2}$, listing the calculated values for each ω.

$$ome[\omega\_] = \frac{1}{-1+\omega^2}$$

$$\frac{1}{-1+\omega^2}$$

```
m = Table[ome[ω], {ω, {0.5, 0.9, 1.1, 1.5, 10}}]
```

$\{-1.33333, -5.26316, 4.7619, 0.8, \frac{1}{99}\}$

```
N[TableForm[{{0.5, -1.3333333333333333`}, {0.9, -5.263157894736843`},
   {1.1, 4.761904761904757`}, {1.5, 0.8`}, {10, 1/99}},
  TableHeadings → {{}, {"ω", "m[ω]"}}]]
```

ω	m[ω]
0.5	-1.33333
0.9	-5.26316
1.1	4.7619
1.5	0.8
10.	0.010101

The above matches the text, though the table construction seemed more time-consuming than profitable.

11. $r(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ 1 & \text{if } 0 < t < \pi \end{cases} \quad |\omega| \neq 1, 3, 5, \ldots$

```
Clear["Global`*"]
```

```
r[t_] = Piecewise[{{-1, -π < t < 0}, {1, 0 < t < π}}]
```

$\begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \\ 0 & \text{True} \end{cases}$

First r[t] is considered by finding its Fourier series.

```
e3 = ExpToTrig[
  FourierSeries[Piecewise[{{-1, -π < t < 0}, {1, 0 < t < π}}], t, 6]]
```

$\frac{4\,\text{Sin}[t]}{\pi} + \frac{4\,\text{Sin}[3\,t]}{3\,\pi} + \frac{4\,\text{Sin}[5\,t]}{5\,\pi}$

The above doesn't look bad at all. The general term is $\frac{4}{n\pi}$ Sin[n t],
with n = 1, 3, 5 ... In the text example,
the general term of the Fourier series is set equal to the ODE without apology,
so I will do it too. At this point in the problem,
I am supposed to switch over to considering the ODE,
including that series general term for r[t].

```
eq1 = FullSimplify[DSolve[y''[t] + ω^2 y[t] == 4/(n π) Sin[n t], y[t], t]]
```

$\{\{y[t] \to C[1]\,\text{Cos}[t\,\omega] - \frac{4\,\text{Sin}[n\,t]}{n^3\,\pi - n\,\pi\,\omega^2} + C[2]\,\text{Sin}[t\,\omega]\}\}$

```
eq11 = eq1 /. n → 1
```

$$\left\{\left\{y[t] \to C[1]\,\text{Cos}[t\,\omega] - \frac{4\,\text{Sin}[t]}{\pi - \pi\,\omega^2} + C[2]\,\text{Sin}[t\,\omega]\right\}\right\}$$

```
eq13 = eq1 /. n → 3
```

$$\left\{\left\{y[t] \to C[1]\,\text{Cos}[t\,\omega] - \frac{4\,\text{Sin}[3\,t]}{27\,\pi - 3\,\pi\,\omega^2} + C[2]\,\text{Sin}[t\,\omega]\right\}\right\}$$

```
eq15 = eq1 /. n → 5
```

$$\left\{\left\{y[t] \to C[1]\,\text{Cos}[t\,\omega] - \frac{4\,\text{Sin}[5\,t]}{125\,\pi - 5\,\pi\,\omega^2} + C[2]\,\text{Sin}[t\,\omega]\right\}\right\}$$

This seemed to be going so well. But I could not (quite) get to the text answer. The yellow cells should show the text answer, but the central term of the text answer presents the model $\frac{4}{\pi}\,\frac{\text{Sin}[n\,t]}{\omega^2-(4\,n-1)^2}$, instead of the yellow $\frac{4}{n\,\pi}\,\frac{\text{Sin}[n\,t]}{n^2-\omega^2}$, and I don't understand this result. I checked the integration in Symbolab, and it agreed with *Mathematica* as far as the integration is concerned. Certainly it is possible the text answer is correct.

13 - 16 Steady-State Damped Oscillations

Find the steady-state oscillations of y"+cy'+y = r(t) with c>0 and r(t) as given. Note that the spring constant is k=1. Show the details. In probs. 14 - 16 sketch r(t).

13. r(t) $= \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt)$

```
Clear["Global`*"]
```

Here r[t] is already a series. `r[t_] = ∑_{n=1}^{N} (a Cos[n t] + b Sin[n t])`. Using a method seen in the solutions manual, I will drop the subscripts of the coefficients a and b. (This problem is being solved after finishing problem 15, for which s.m. assistance was available.) I will consider r[t] to be a single term of the series.

```
r[t_] = a Cos[n t] + b Sin[n t]
a Cos[n t] + b Sin[n t]
```

```
r'[t]
b n Cos[n t] - a n Sin[n t]
```

```
r''[t]
-a n^2 Cos[n t] - b n^2 Sin[n t]
```

```
partic = r''[t] + c r'[t] + r[t]
```

$a \,\text{Cos}[n\, t] - a\, n^2 \,\text{Cos}[n\, t] + b \,\text{Sin}[n\, t] - b\, n^2 \,\text{Sin}[n\, t] + c\, (b\, n \,\text{Cos}[n\, t] - a\, n \,\text{Sin}[n\, t])$

```
eq1 = Simplify[partic]
```

$\left(a + b\, c\, n - a\, n^2\right) \text{Cos}[n\, t] + \left(b - a\, c\, n - b\, n^2\right) \text{Sin}[n\, t]$

For this problem, evidently the RHS will have both sine and cosine terms. The value of N is unknown, but it could encompass any number of 2π cycles. The coefficients must keep the same ratios at all points of the trig circle, so I take the guess that A_n will be solved when the function is at zero (cosine function is max), and B_n will be solved when the function is at $\pi/2$ (sine function is max). So eq2 will be for A_n:

```
eq2 = Solve[{a + b * c * n - a * n^2 == 1, b - a * c * n - b * n^2 == 0}, {a, b}]
```

$$\left\{\left\{a \to -\frac{-1 + n^2}{1 - 2\, n^2 + c^2\, n^2 + n^4},\ b \to \frac{c\, n}{1 - 2\, n^2 + c^2\, n^2 + n^4}\right\}\right\}$$

To assemble A_n I suppose that all I need to do is multiply the numerators by the relevant coefficients and add these two together. (I can already check the 'D_n' value, the denominator, with the text and confirm that it agrees.)

$$\text{bigA} = \text{Simplify}\left[-\frac{\left(-1 + n^2\right) \text{asubn}}{1 - 2\, n^2 + c^2\, n^2 + n^4} + \frac{(c\, n)\ \text{bsubn}}{1 - 2\, n^2 + c^2\, n^2 + n^4}\right]$$

$$\frac{\text{asubn} + \text{bsubn}\ c\, n - \text{asubn}\ n^2}{1 + \left(-2 + c^2\right) n^2 + n^4}$$

The method works for A_n above, which agrees with the text. Now to try to figure out B_n, which I predict must come into alignment at trig $\pi/2$:

```
eq3 = Solve[{a + b * c * n - a * n^2 == 0, b - a * c * n - b * n^2 == -1}, {a, b}]
```

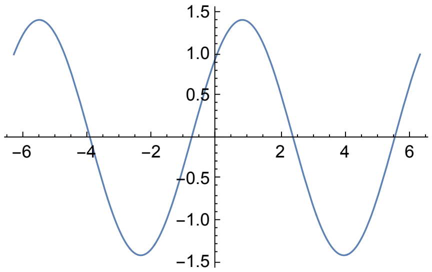
$$\left\{\left\{a \to \frac{c\, n}{1 - 2\, n^2 + c^2\, n^2 + n^4},\ b \to -\frac{1 - n^2}{1 - 2\, n^2 + c^2\, n^2 + n^4}\right\}\right\}$$

$$\text{BigB} = \text{Simplify}\left[\frac{(c\, n)\ \text{asubn}}{1 - 2\, n^2 + c^2\, n^2 + n^4} - \frac{\left(-1 + n^2\right) \text{bsubn}}{1 - 2\, n^2 + c^2\, n^2 + n^4}\right]$$

$$\frac{\text{bsubn} + \text{asubn}\ c\, n - \text{bsubn}\ n^2}{1 + \left(-2 + c^2\right) n^2 + n^4}$$

The method works for B_n too, *except* that in order to get the sign of a_n to agree with the text, it was necessary to choose -$\pi/2$ as the point of evaluation, so that the a_n part of the B_n ensemble could be positive in sign. I don't know how to interpret that requirement.

```
Plot[ Cos[ t] + Sin[ t], {t, -2 π, 2 π}, PlotStyle → Thickness[0.004]]
```



The plot (above) does not look quite as expected. I feel I should emphasize that the described solution method is largely speculation.

15. r(t) = $t(\pi^2 - t^2)$ if -π<t<π , and r(t+2π) = r(t)

This problem is covered in the s.m.. The observation, made there and visible from the problem description, is that the function r[t] is odd and that the function's cycle is 2π. At this point I check the Fourier series.

```
Clear["Global`*"]

eq1 = FourierSeries[t (π² - t²), t, 1]
6 ⅈ ⅇ^(-ⅈ t) - 6 ⅈ ⅇ^(ⅈ t)

eq2 = ExpToTrig[6 ⅈ ⅇ^(-ⅈ t) - 6 ⅈ ⅇ^(ⅈ t)]
12 Sin[t]
```

So at this point I know the form of the output series. No cosine term. I don't take the '12' too seriously, it is still subject to some variation.

The method of finding a particular solution in example 1 on p. 493 sees it as y"+-cy'+y=b_n sin nt. Here the s.m. makes reference to example 1, where in a similar situation the y_p is set to y= A cos nt + B sin nt. The motivation for this is an entry in Table 2.1, p. 82, "*Method of Undetermined Coefficients*, where, upon finding r[t] equal to k sin ωx, a preliminary choice for $y_p(x)$ is taken as K cos ωx +M sin ωx. So at this point I have [1]: y=A cos nt + B sin nt, and I go on to assign [2]: y'=-A sin nt + B cos nt, and also [3]: y"=-A cos nt -B sin nt.

```
partic = (y''[t] + c y'[t] + y[t])
y[t] + c y′[t] + y″[t]
```

partic is the LHS

```
r[t] = A Cos[n t] + B Sin[n t] +
  c (-n A Sin[n t] + n B Cos[n t]) - n² A Cos[n t] - n² B Sin[n t]
A Cos[n t] - A n² Cos[n t] + B Sin[n t] -
 B n² Sin[n t] + c (B n Cos[n t] - A n Sin[n t])
```

r[t] is the consolidation of plugging values of the 3 equations into LHS and adding them up.

```
Simplify[r[t]]
```

$(A + B c n - A n^2) \text{Cos}[n t] + (B - A c n - B n^2) \text{Sin}[n t]$

Now it is time to solve for coefficients of the r[t] complex. Final coefficient of cosine must be zero (since it doesn't appear in final r) and final coefficient of sine must be b_n. As for n, it can vary in series fashion. It is necessary to humor Mathematica a bit, as for instance not using variables beginning with captitals, and, for just this once, eschewing subscripts (m is standing in for b_n);

$\text{eq3} = \text{Solve}[\{a + b * c * n - a * n^2 == 0, b - a * c * n - b * n^2 == m\}, \{a, b\}]$

$$\left\{\left\{a \to -\frac{c\, m\, n}{1 - 2 n^2 + c^2 n^2 + n^4}, b \to -\frac{m(-1 + n^2)}{1 - 2 n^2 + c^2 n^2 + n^4}\right\}\right\}$$

Solve does the solve thing, and sets the denominator to the correct value of $D_{n.}$ In the cell below, it will be done in the determinant way.

$$\text{dee} = \text{Det}\left[\begin{pmatrix} 1 - n^2 & c\, n \\ -c\, n & 1 - n^2 \end{pmatrix}\right]$$

$1 - 2 n^2 + c^2 n^2 + n^4$

The s.m. now goes on to find A and B, using determinants, but will it thereby find what **Solve** came up with above? The current step is to find $b_{n,}$ which Mathematica has not yet found, and which it cannot find by modifying eq3 for the search. But the s.m. goes back to a table on page 487, where is says that an odd function with period 2π should follow the formula $b_n = \frac{2}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$ and $n = 1, 2, \ldots$ Okay, I'll follow.

$$\text{bn} = \frac{2}{\pi} \text{Integrate}[t(\pi^2 - t^2) \text{Sin}[n t], \{t, 0, \pi\}]$$

$$\frac{2(-6 n \pi \text{Cos}[n \pi] - 2(-3 + n^2 \pi^2) \text{Sin}[n \pi])}{n^4 \pi}$$

$\text{int} = \text{bn} \;/.\; \text{Cos}[n \pi] \to (-1)^n$

$$\frac{2(-6 (-1)^n n \pi - 2(-3 + n^2 \pi^2) \text{Sin}[n \pi])}{n^4 \pi}$$

$b_n = \text{int} \;/.\; \text{Sin}[n \pi] \to 0$

$$-\frac{12 (-1)^n}{n^3}$$

With two invaluable trig substitutions provided by s.m., b_n is determined, above, green. I now have the value of 'm' in eq3, and I want to use it to find the total A, using the numerator of the ‘a’ part of eq3.

```
aaa = -cn (b_n)
```

$-cn\, b_n$

```
aaaa = aaa /. b_n -> - 12 (-1)^n / n^3
```

$$\frac{12\,(-1)^n\, cn}{n^3}$$

```
aaaaa = aaaa / dee
```

$$\frac{12\,(-1)^n\, cn}{n^3\left(1 - 2 n^2 + c^2 n^2 + n^4\right)}$$

Above is the final value of A, which agrees with the text answer.

```
bbb = - (-1 + n^2) b_n
```

$\left(1 - n^2\right) b_n$

```
bbbb = bbb /. b_n → - 12 (-1)^n / n^3
```

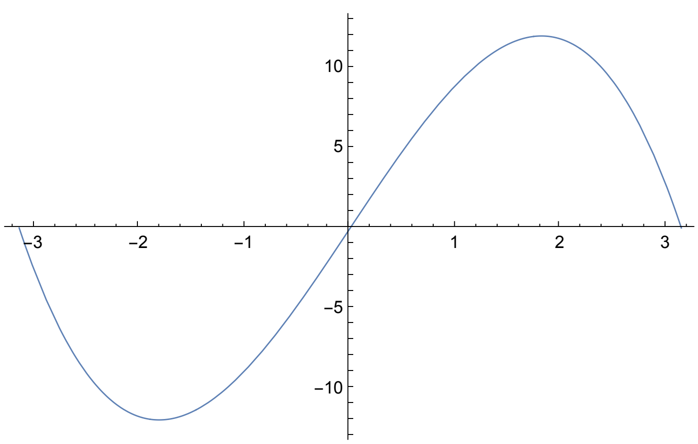
$$-\frac{12\,(-1)^n\left(1 - n^2\right)}{n^3}$$

```
bbbbb = bbbb / dee
```

$$-\frac{12\,(-1)^n\left(1 - n^2\right)}{n^3\left(1 - 2 n^2 + c^2 n^2 + n^4\right)}$$

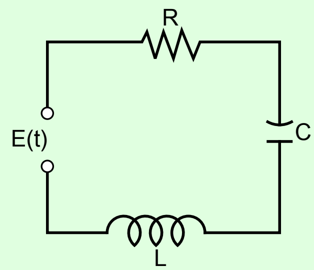
Above is the final answer of B, which agrees with the text answer. (Note that $(-1)^n$ resolves to $(-1)^{n+1}$.) This problem also requires a sketch of r[t].

```
rtplot = Plot[t (π^2 - t^2), {t, -π, π}, PlotStyle → Thickness[0.002]]
```



17 - 19 RLC-circuit.

Find the steady state current I(t) in the RLC-circuit in figure 275, where R=10 Ω, L=1 H, C=10^{-1} F and with E(t) V as follows and periodic with period 2π. Graph or sketch the first four partial sums. Note that the coefficients of the solution decrease rapidly. Hint. Remember that the ODE contains $E'(t)$, not $E(t)$, cf. section 2.9.



RLC–circuit (after Fig. 275)

17. `E[t] = Piecewise[{{-50 t², -π < t < 0}, {50 t², 0 < t < π}}]`

```
In[80]:= Clear["Global`*"]
```

First, setting up the electrical state space model just as if the domain were not piecewise.

```
In[81]:= eqns = {eL q''[t] + aR q'[t] + 1/cC q[t] == Vee[t]};

In[84]:= m1 = StateSpaceModel[eqns,
           {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

Out[84]=
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{cC\ eL} & -\frac{aR}{eL} & \frac{1}{eL} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

And putting in the capacitance, inductance, and resistance from the problem description.

```
In[85]:= mw = m1 /. {cC → 0.1, eL → 1, aR → 10}
```

Out[85]=
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -10. & -10 & 1 \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

And getting an output response for the interval where the voltage is negative. Note that this is in an interval where t is negative. What does a negative time value represent? I don't really blame Mathematica for dumping the output into a single point, probably zero.

```
In[86]:= outz = OutputResponse[{mw}, -50 t², {t, -π, 0}]

Out[86]= {InterpolatingFunction[ Domain: {{0., 0.}} Output: scalar ][t]}
```

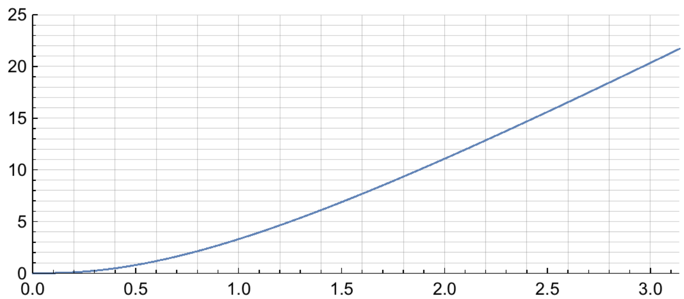
And another output response for the interval where the voltage (and time) is positive.

In[90]:=
```
outzp = OutputResponse[{mw}, 50 t^2, {t, 0, π}]
```

Out[90]=
```
{InterpolatingFunction[ Domain: {{0., 3.14}} Output: scalar ][t]}
```

And devising a plot for the positive t section. If this was a serious project, I would look into the possibility of moving everything to the right so that all time values would be positive.

In[92]:=
```
p1 = Plot[{outzp}, {t, 0, π}, ImageSize → 350,
  AspectRatio → 0.4, PlotRange → {{-0.001, π}, {-0.005, 25}},
  PlotStyle → Thickness[0.003], GridLines → All]
```

Out[92]=


In[21]:=
```
NIntegrate[outzp, {t, 0, 3}]
```

Out[21]=
```
{23.6933}
```

$$an = \frac{-400}{(n^2 \pi)}$$

$$-\frac{400}{n^2 \pi}$$

$$dn = (n^2 - 10)^2 + 100\, n^2$$

$$100\, n^2 + (-10 + n^2)^2$$

$$han = N\left[Table\left[\frac{(10 - n^2)\, an}{dn}, \{n, 1, 6\}\right]\right]$$

```
{-6.33103, -0.438041, -0.0157016, 0.0291849, 0.0280346, 0.0215052}
```

$$hbn = N\left[Table\left[\frac{(10\, n)\, an}{dn}, \{n, 1, 6\}\right]\right]$$

```
{-7.03447, -1.46014, -0.471047, -0.194566, -0.0934488, -0.0496274}
```

```
eye = N[
  50 + han[[1]] Cos[3] + hbn[[1]] Sin[3] + han[[3]] Cos[9] + hbn[[3]] Sin[9]]
55.0951
```

There is something obviously wrong with this problem somewhere. There is an absurdly large gap between what I am coming up (for steady state current) using the text answer and when using the state space model. This is in gross contrast to the results in section 2.9, when the state space model was on the money. However, back then the function went into cycles, whereas here it just rises. The problem description describes the function as 2π periodic, but I see no tendency to operate in a cycle.

19. **E[t] = Piecewise[{200 t (π² t²), -π < t < π}, {0, -∞ < t ≤ -π}]**

In[68]:=
```
Clear["Global`*"]
```

First, setting up the electrical state space model just as if the domain were not piecewise.

In[69]:=
```
eqns = {eL q''[t] + aR q'[t] + 1/cC q[t] == Vee[t]};
```

In[70]:=
```
m1 = StateSpaceModel[eqns,
  {{q[t], 0}, {q'[t], 0}}, {{Vee[t], 0}}, {q'[t]}, t]
```

Out[70]=
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -\frac{1}{cC\,eL} & -\frac{aR}{eL} & \frac{1}{eL} \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

And putting in the capacitance, inductance, and resistance from the problem description.

In[71]:=
```
mw = m1 /. {cC → 0.1, eL → 1, aR → 10}
```

Out[71]=
$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -10. & -10 & 1 \\ \hline 0 & 1 & 0 \end{array}\right)^{\mathcal{S}}$$

And getting an output response for the interval where the voltage is non-zero. Note that part of this interval occurs where t is negative. What does a negative time value represent? The way Mathematica handles this situation is to ignore the negative time interval.

In[72]:=
```
outz = OutputResponse[{mw}, 200 t (π² t²), {t, -π, π}]
```

Out[72]=
```
{InterpolatingFunction[ Domain: {{0., 3.14}}  Output: scalar ][t]}
```

In[73]:=
```
NIntegrate[outz, {t, 0, 3}]
```

Out[73]=
```
{2282.5}
```

And another output response for the interval where the voltage is zero and the time is positive.

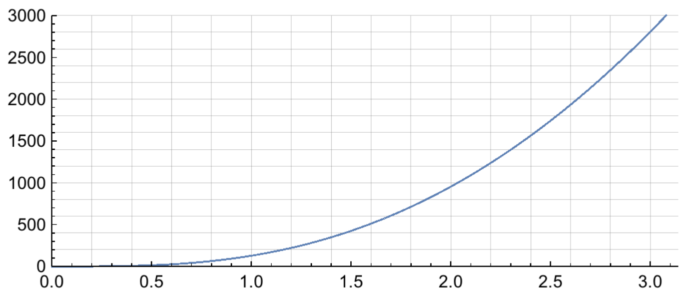
In[51]:=
```
outz0 = OutputResponse[{mw}, 0, {t, π, 20}]
```

Out[51]=
```
{InterpolatingFunction[ Domain: {{0., 20.}}  Output: scalar ][t]}
```

And devising a plot for the positive t section. If this was a serious project, I would look into the possibility of moving everything to the right so that all time values would be positive. The plot below looks reasonable to me.

In[38]:=
```
p1 = Plot[{outz}, {t, 0, π}, ImageSize → 350,
  AspectRatio → 0.4, PlotRange → {{-0.001, π}, {-0.005, 3000}},
  PlotStyle → Thickness[0.003], GridLines → All]
```

Out[38]=



In[61]:= $D_n = \left(10 - n^2\right)^2 + 100\,n^2$

Out[61]= $100\,n^2 + \left(10 - n^2\right)^2$

In[62]:= $A_n = (-1)^{n+1}\,\frac{2400\left(10 - n^2\right)}{n^2\,D_n}$

Out[62]= $\frac{2400\,(-1)^{1+n}\left(10 - n^2\right)}{n^2\left(100\,n^2 + \left(10 - n^2\right)^2\right)}$

In[63]:= $B_n = (-1)^{n+1}\,\frac{24\,000}{n\,D_n}$

Out[63]= $\frac{24\,000\,(-1)^{1+n}}{n\left(100\,n^2 + \left(10 - n^2\right)^2\right)}$

In[77]:=
```
eye = N[
   Simplify[ComplexExpand[Re[Sum[Aₙ Cos[n t] + Bₙ Sin[n t], {n, 1, ∞}]]]]];
```

In[66]:=
```
eyet3 = eye /. t → 3
```

Out[66]= -89.8515

Way off base. See the comments in problem 17.